PITHAPUR RAJAH'S GOVERNMENT COLLEGE (AUTONOMOUS), KAKINADA

DEPARTMENT OF MATHEMATICS

Differential Equations

Syllabus:

UNIT - I **(12 Hours)**

Differential Equations of first order and first degree: Linear Differential Equations; Differential equations reducible to linear form (Bernoulli's Differential Equations); Exact differential equations; Integrating factors.

Definition:(Differential Equations of first order and first degree)

A differential equation is in the form $\frac{dy}{dx} = f(x, y)$ is called a differential equation of the first order and of the first degree.

We study the following methods for solving $\frac{dy}{dx} = f(x, y)$

1. Variable separable. (Intermediate)
2. Homogeneneous differential equations. (Intermediate)
3. Non-Homogeneous differential equations. (Intermediate)
4. Linear differential equations.
5. Bernoullis Differential equations.
6. Exact Differential Equations.
7. Non- Exact Differential Equations.

Linear differential equations:

A differential equation is in the form $\frac{dy}{dx} + P\,y = Q$ where P and Q are functions of x ,is called Linear differential equation.

Working Rule to solve the equation:

1. First to reduce the given differential equation in the form $\frac{dy}{dx} + P\,y = Q$

   and then identify P and Q.

2. To find the Integrating Factor (I.F) = $e^{\int P\,dx}$

3. And then to find the general solution $y(I.F) = \int Q(I.F)dx + c.$

Problems:

1. Solve $\frac{dy}{dx} + 2xy = e^{-x^2}$

Solution: Given differential equation $\frac{dy}{dx} + 2xy = e^{-x^2}$

Let P = $2x$ & Q = e^{-x^2}

The integrating factor (I.F) = $e^{\int P\,dx} = e^{\int 2x\,dx} = e^{2(\frac{x^2}{2})} = e^{x^2}$

$$\text{I F} = e^{x^2}$$

The general solution is $y(I.F) = \int Q(I.F)dx + c.$

$$y(e^{x^2}) = \int e^{-x^2}(e^{x^2})dx + c = \int 1dx + c = x + c$$

$$\therefore ye^{x^2} = x + c$$

2. Solve $x \log x \frac{dy}{dx} + y = 2\log x$

Solution: Given differential equation $x \log x \frac{dy}{dx} + y = 2\log x$

It can be reduced to $\frac{dy}{dx} + \frac{1}{x\log x}\ y = \frac{2}{x}$

Let P = $\frac{1}{x\log x}$ & Q = $\frac{2}{x}$

The integrating factor (I.F) = $e^{\int P\,dx} = e^{\int \frac{1}{x\log x}dx} = e^{\log(\log x)} = \log x$

I F = $\log x$

The general solution is $y(I.F) = \int Q(I.F)dx.$ $\quad let\ log\ x = t \Rightarrow \frac{1}{x}\,dx = dt$

$$y\log x = \int \frac{2}{x}(\log x)dx = 2\int t\,dt = 2\left(\frac{t^2}{2}\right) + c = t^2 + c = (\log x)^2 + c$$

∴ The general solution $y\log x = (\log x)^2 + c$

3. Solve $x\frac{dy}{dx} + 2y = x^2\log x$

Solution: Given differential equation $x\frac{dy}{dx} + 2y = x^2\log x$

It can be reduced to $\frac{dy}{dx} + \frac{2}{x}\ y = x\log x$

Let P = $\frac{2}{x}$ & Q = $x\log x$

The integrating factor (I.F) = $e^{\int P\,dx} = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$

I F = x^2

The general solution is $y(I.F) = \int Q(I.F)dx + c.$

$$yx^2 = \int x\log x\ (x^2)dx = \int x^3\log x\ dx + c$$

$$= \int \log x\ (x^3)\ dx + c$$

$$= (\,log\ x)\frac{x^4}{4} - \int \frac{1}{x}\ \left(\frac{x^4}{4}\right)dx + c$$

$$= \frac{x^4}{4}(\,log\ x) - \frac{1}{4}\int x^3\,dx + c$$

$$yx^2 = \frac{x^4}{4}(\,log\ x) - \frac{1}{4}\ \frac{x^4}{4} + c$$

$$= \frac{x^4}{4}(\log x) - \frac{x^4}{16} + c$$

$\therefore$ The general solution $yx^2 = \frac{x^4}{4}(\log x) - \frac{x^4}{16} + c$

4. Solve $(x^2+1)\frac{dy}{dx} + 4xy = \frac{1}{x^2+1}$

Solution: Given differential equation $(x^2+1)\frac{dy}{dx} + 4xy = \frac{1}{x^2+1}$

It can be reduced to $\frac{dy}{dx} + \frac{4x}{(x^2+1)}\,y = \frac{1}{(x^2+1)^2}$

Let P = $\frac{4x}{(x^2+1)}$ & Q = $\frac{1}{(x^2+1)^2}$

The integrating factor (I.F) = $e^{\int P\,dx} = e^{\int \frac{4x}{(x^2+1)}dx} = e^{2\int \frac{2x}{(x^2+1)}dx}$

$$= e^{2\log(x^2+1)} = e^{\log(x^2+1)^2}$$

$$I\,F = (x^2+1)^2$$

The general solution is $y(I.F) = \int Q(I.F)dx + c.$

$$y(x^2+1)^2 = \int (x^2+1)^2 \frac{1}{(x^2+1)^2}dx + c = \int 1\,dt + c = x + c$$

$\therefore$ The general solution $y(x^2+1)^2 = x + c$

5. Solve $(x+1)\frac{dy}{dx} - xy = 1 - x$

Solution: Given differential equation $(x+1)\frac{dy}{dx} - xy = 1 - x$

It can be reduced to $\frac{dy}{dx} - \frac{x}{(1+x)}\,y = \frac{1-x}{1+x}$

Let P = $-\frac{x}{(1+x)}$ and Q = $\frac{1-x}{1+x}$

The integrating factor (I.F) = $e^{\int P\,dx} = e^{-\int \frac{x}{(1+x)}dx} = e^{-\int \frac{1+x-1}{(1+x)}dx}$

$$= e^{-\int 1 - \frac{1}{(1+x)}dx}$$

$$= e^{-[x - \log(1+x)]}$$

$$= e^{-[x - \log(1+x)]}$$

$$= e^{-x}\ e^{\log(1+x)} = (1+\ x)\ e^{-x}$$

$$\text{I F} = (1+\ x)\ e^{-x}$$

The general solution is $y(I.F) = \int Q(I.F)dx + c.$

$$y(1+\ x)e^{-x} = \int\ (1+\ x)e^{-x}\ \left[\frac{1-x}{1+x}\right] dx + c = \int(1-x)e^{-x}\ dt + c$$

$$= (1-x)(-e^{-x}) - \int(-1)(-e^{-x})dx + \text{c}$$

$$= -(1-x)e^{-x} - \int e^{-x}dx\ + c$$

$$y(1+\ x)e^{-x} = -(1-x)e^{-x} + e^{-x} + c = xe^{-x} + c$$

$\therefore$ The general solution $y(1+\ x)e^{-x} = xe^{-x} + c$

6. Solve $x\frac{dy}{dx} + ylog\ x = e^{x}\ x^{1-\frac{1}{2}\log x}$

Solution: Given differential equation $x\frac{dy}{dx} + ylog\ x = e^{x}\ x^{1-\frac{1}{2}\log x}$

It can be reduced to $\frac{dy}{dx} + y\ \frac{logx}{x} = \frac{e^{x}\ x^{1-\frac{1}{2}\log x}}{x} = e^{x}\ x^{-\frac{1}{2}\log x}$

Let P = $\frac{logx}{x}$ & Q = $e^{x}\ x^{-\frac{1}{2}\log x}$

The integrating factor (I.F) = $e^{\int P\,dx} = e^{\int \frac{logx}{x} dx}$

$$= e^{\frac{(\log x)^2}{2}}$$

$$= e^{\log x\,\frac{\log x}{2}} = [\,e^{\log x}]^{\frac{logx}{2}} = x^{\frac{1}{2}\log x}$$

$$\text{I F} = x^{\frac{1}{2}\log x}$$

The general solution is $y(I.F) = \int Q(I.F)dx + c.$

$$y\,x^{\frac{1}{2}\log x} = \int e^{x}\ x^{-\frac{1}{2}\log x}\ (x^{\frac{1}{2}\log x})\,dx + \text{c}$$

$$= \int e^{x}\ d\,x\ + c = e^{x} + c$$

The general solution $y\,x^{\frac{1}{2}\log x} = e^{x} + c$

7. Solve $x^2\frac{dy}{dx}+(x-2)y=x^2\ e^{-\frac{2}{x}}$

Solution: Given differential equation $x^2\frac{dy}{dx}+(x-2)y=x^2\ e^{-\frac{2}{x}}$

It can be reduced to $\frac{dy}{dx}+\frac{1}{x^2}(x-2)y=\ e^{-\frac{2}{x}}$

Let P $=\frac{x-2}{x^2}=\frac{1}{x}-\frac{2}{x^2}$ & Q $=\ e^{-\frac{2}{x}}$

The integrating factor (I.F) $= e^{\int P\,dx}=e^{\int[\frac{1}{x}-\frac{2}{x^2}]dx}=e^{\log x+\frac{2}{x}}$

$$= e^{\log x}\,e^{\frac{2}{x}}=x\,e^{\frac{2}{x}}$$

I F $=x\,e^{\frac{2}{x}}$

The general solution is $y(I.F)=\int Q(I.F)dx+c.$

$$y\ x\,e^{\frac{2}{x}}=\int x\,e^{\frac{2}{x}}\,e^{-\frac{2}{x}}dx+c=\int x\,dt+c\ =\frac{x^2}{2}+c$$

$\therefore$ The general solution $y\ x\,e^{\frac{2}{x}}=\frac{x^2}{2}+c$

8.. Solve $(\ 1+x^2)\frac{dy}{dx}+2xy=4\,x^2$

Solution: Given differential equation $(\ 1+x^2)\frac{dy}{dx}+2xy=4\,x^2$

It can be reduced to $\frac{dy}{dx}+\frac{2x}{(1+x^2)}\ y=\frac{4\,x^2}{1+x^2}$

Let P $=\frac{2x}{(1+x^2)}$ & Q $=\frac{4\,x^2}{1+x^2}$

The integrating factor (I.F) $= e^{\int P\,dx}=e^{\int\frac{2x}{(x^2+1)}dx}=e^{\log(1+x^2)}\ =\ 1+x^2$

I F $=\ 1+x^2$

The general solution is $y(I.F)=\int Q(I.F)dx+c.$

$$y(\ 1+\ x^2)=\int(\ 1+\ x^2)[\frac{4\,x^2}{1+x^2}]\ dx+c=\int 4x^2\ dt+c\ =\frac{4x^3}{3}+c$$

$\therefore$ The general solution $y(\ 1+\ x^2)=\frac{4x^3}{3}+c$

9. Solve $cos^2x \frac{dy}{dx} + y = \tan x$

Solution: Given differential equation $cos^2x \frac{dy}{dx} + y = \tan x$

It can be reduced to $\frac{dy}{dx} + \frac{1}{cos^2x} y = \frac{Tan\, x}{cos^2x} \Rightarrow \frac{dy}{dx} + sec^2x \ \ y = \tan x \ \ sec^2x$

Let P = sec^2x & Q = $\tan x \ \ sec^2x$

The integrating factor (I.F) = $e^{\int P\, dx} = e^{\int sec^2x\, dx} = e^{Tan\, x}$

$$\text{I F} = e^{Tan\, x}$$

The general solution is $y(I.F) = \int Q(I.F)dx + c.$

$$ye^{Tan\, x} = \int \tan x \ sec^2x \ e^{Tan\, x} dx + c$$

$$\text{Let } \tan x = t \ \Rightarrow sec^2 x\, dx = dt$$

$\therefore \ ye^{Tan\, x} = \int t\, e^t \ dt \ +c = e^t(t-1) + c$

$\therefore$ The general solution $ye^{Tan\, x} = e^{Tan\, x}(\tan x - 1) + c$

10. Solve $(x + 2y^3) \frac{dy}{dx} = y.$

Solution: To convert the differential equation in the form $\frac{dx}{dy} + P\, x = Q$

Given that $(x + 2y^3) \frac{dy}{dx} = y \Rightarrow y \frac{dx}{dy} = x + 2y^3$

$$\Rightarrow y \frac{dx}{dy} - x = 2y^3$$

$$\Rightarrow \frac{dx}{dy} - \frac{x}{y} = 2y^2$$

$$Let\ P = \frac{-1}{y} \quad and\ Q = 2y^2$$

The integrating factor (I.F) = $e^{\int P\, dy} = e^{\int \frac{-1}{y} dy} = e^{-\log y} = \frac{1}{y}$ $\therefore$ I F = $\frac{1}{y}$

The general solution $x(I.F) = \int Q(I.F)dy + c$

$x\left(\frac{1}{y}\right) = \int 2y^2 \left(\frac{1}{y}\right) dy + c = \int 2y dy + c = \frac{2y^2}{2} + c = y^2 + c$

Hence the general solution $\frac{x}{y} = y^2 + c$

11.Solve $(1+y^2)dx = (\tan^{-1} y - x)dy$.

Solution: To convert the differential equation in the form $\frac{dx}{dy} + P\,x = Q$

Given that $(1+y^2)dx = (\tan^{-1} y - x)dy$

$$\Rightarrow (1+y^2)\frac{dx}{dy} = (\tan^{-1} y - x)$$

$$\Rightarrow (1+y^2)\frac{dx}{dy} + x = \tan^{-1} y \Rightarrow \frac{dx}{dy} + \frac{1}{1+y^2}\,x = \frac{\tan^{-1} y}{1+y^2}$$

$$Let\ P = \frac{1}{1+y^2} \quad and\ Q = \frac{\tan^{-1} y}{1+y^2}$$

The integrating factor (I.F) = $e^{\int P\,dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1} y}$ $\therefore$ I F = $e^{\tan^{-1} y}$

The general solution $x(I.F) = \int Q(I.F)dy + c$

$x(\ e^{\tan^{-1} y}) = \int \frac{\tan^{-1} y}{1+y^2}(\ e^{\tan^{-1} y})dy + c$ ------(1)

Let $\tan^{-1} y = t \Rightarrow \frac{1}{1+y^2}dy = dt$

(1) Can be written as $x(\ e^{\tan^{-1} y}) = \int te^t dt + c = e^t(t-1) + c$

Hence general solution is $x(\ e^{\tan^{-1} y}) = e^{\tan^{-1} y}(\tan^{-1} y - 1) + c$

Bernoulli's Differential equations:

A differential equation is in the form $\frac{dy}{dx} + P\,y = Qy^n$ where P and Q are functions of x ,is called Bernoulli's differential equation.

Working Rule to solve the equation:

1.First to divide the DE with y^n *on both sides* $\frac{1}{y^n}\frac{dy}{dx} + P\,y^{1-n} = Q$ -----$(*)$

2. Put $y^{1-n} = z$ and then $(1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \frac{1}{y^n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx}$

3.Substitute the values in $(*)$

$$\frac{1}{1-n}\frac{dz}{dx} + P\,z = Q \text{ is a linear differential equation and then solve.}$$

Problems:

1. Solve $\frac{dy}{dx}+\frac{y}{x}=x^2y^6$

Solution: The given differential equation $\frac{dy}{dx}+\frac{y}{x}=x^2y^6$

Divide the DE with y^6 *on both sides* $\frac{1}{y^6}\frac{dy}{dx}+\frac{y^{-5}}{x}=x^2$ ------------ (1)

Put $y^{-5}=z$ and then $-5y^{-5-1}\frac{dy}{dx}=\frac{dz}{dx} \Rightarrow \frac{1}{y^6}\frac{dy}{dx}=\frac{1}{-5}\frac{dz}{dx}$

(1) Can be written as $\frac{-1}{5}\frac{dz}{dx}+\frac{z}{x}=x^2 \Rightarrow \frac{dz}{dx}-\frac{5z}{x}=-5x^2$

Let $P_1=\frac{-5}{x}$ *and* $Q_1=-5x^2$

The integrating factor (I.F) = $e^{\int P_1\,dx}$

$$=e^{\int\frac{-5}{x}dx}=e^{-5\int\frac{1}{x}dx}$$

$$=e^{-5\log x}=e^{\log x^{-5}}=x^{-5}=\frac{1}{x^5} \qquad I\,F=\frac{1}{x^5}$$

The general solution: $z(I.F)=\int Q_1(I.F)dx+c$

$$y^{-5}\frac{1}{x^5}=\int -5x^2\left(\frac{1}{x^5}\right)dx+c$$

$$\frac{1}{x^5y^5}=-5\int x^{-3}dx+c=-5\left[\frac{x^{-3+1}}{-3+1}\right]+c$$

The general solution $\frac{1}{x^5y^5}=\frac{5}{2x^2}+c$

2. Solve $\frac{dy}{dx}+\frac{y}{x}=y^2x\sin x$

Solution: The given differential equation $\frac{dy}{dx}+\frac{y}{x}=y^2x\sin x$

Divide the DE with y^2 *on both sides* $\frac{1}{y^2}\frac{dy}{dx}+\frac{y^{-1}}{x}=x\sin x$ -------- (1)

Put $y^{-1}=z$ and then $(-1)\,y^{-1-1}\frac{dy}{dx}=\frac{dz}{dx} \Rightarrow \frac{1}{y^2}\frac{dy}{dx}=-\frac{dz}{dx}$

(1) Can be written as $-\frac{dz}{dx}+\frac{z}{x}=x\sin x \Rightarrow \frac{dz}{dx}-\frac{z}{x}=-x\sin x$

Let $P_1=\frac{-1}{x}$ *and* $Q_1=-x\sin x$

The integrating factor (I.F) = $e^{\int P_1\, dx}$

$$= e^{\int \frac{-1}{x} dx}$$

$$= e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x} \qquad I\,F = \frac{1}{x}$$

The general solution: $z(I.F) = \int Q_1(I.F)dx + c$

$$y^{-1}\ \frac{1}{x} = \int -x \sin x \left(\frac{1}{x}\right) dx + c$$

$$\frac{1}{xy} = -\int \sin x\, dx + c = \cos x + c$$

Hence the GS is $\frac{1}{xy} = \cos x + c$

3. Solve $x\frac{dy}{dx} + y = y^2 \log x$

Solution: The given differential equation $x\frac{dy}{dx} + y = y^2 \log x$

Divide the DE with xy^2 *on both sides* $\frac{1}{y^2}\frac{dy}{dx} + \frac{y^{-1}}{x} = \frac{\log x}{x}$ -------(1)

Put $y^{-1} = z$ and then $(-1)\, y^{-1-1}\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \frac{1}{y^2}\frac{dy}{dx} = -\frac{dz}{dx}$

(1) Can be written as $-\frac{dz}{dx} + \frac{z}{x} = \frac{\log x}{x} \Rightarrow \frac{dz}{dx} - \frac{z}{x} = -\frac{\log x}{x}$

Let $P_1 = \frac{-1}{x}$ *and* $Q_1 = -\frac{\log x}{x}$

The integrating factor (I.F) = $e^{\int P_1\, dx}$

$$= e^{\int \frac{-1}{x} dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x} \qquad \boldsymbol{I\,F = \frac{1}{x}}$$

The general solution: $z(I.F) = \int Q_1(I.F)dx + c$

$$y^{-1}\ \frac{1}{x} = \int -\frac{\log x}{x}\left(\frac{1}{x}\right) dx + c = -\int \frac{1}{x^2} \log x\ dx + c$$

$$\frac{1}{xy} = -\left[\log x\ \left\{-\frac{1}{x}\right\} - \int \frac{1}{x}\left\{-\frac{1}{x}\right\} dx\right] + c$$

$$\frac{1}{xy} = \frac{1}{x}\ \log x - \int \frac{1}{x^2} dx + c = \frac{1}{x}\ \log x + \frac{1}{x} + c$$

Hence the GS is $\frac{1}{xy} = \frac{1}{x}\,[\,\log x + 1] + c$

4. $solve\quad 3\frac{dy}{dx} + \frac{2y}{1+x} = \frac{x^3}{y^2}$

Solution: *Multiply the given differential equation with* $\frac{y^2}{3}$

$$y^2\frac{dy}{dx} + \frac{2y^3}{3(1+x)} = \frac{x^3}{3} \text{-----------------} \quad (1)$$

Put $y^3 = z$ and then $3y^2\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow y^2\frac{dy}{dx} = \frac{1}{3}\frac{dz}{dx}$

(1) Can be written as $\frac{1}{3}\frac{dz}{dx} + \frac{2z}{3(1+x)} = \frac{x^3}{3} \Rightarrow \frac{dz}{dx} + \frac{2z}{(1+x)} = x^3$

Let $P_1 = \frac{2}{1+x}$ *and* $Q_1 = x^3$

The integrating factor (I.F) $= e^{\int P_1\,dx}$

$$= e^{\int \frac{2}{1+x}dx} = e^{2\log(1+x)} = e^{\log(1+x)^2} = (1+x)^2$$

$$\boldsymbol{I\,F} = (1+x)^2$$

The general solution: $z(I.F) = \int Q_1(I.F)dx + c$

$$y^3\,(1+x)^2 = \int x^3(1+x)^2dx + c$$

$$= \int x^3(1 + 2x + x^2)\,dx$$

$$= \int\ [x^3 + 2x^4 + x^5]dx = \frac{x^4}{4} + \frac{2x^5}{5} + \frac{x^6}{6} + c$$

Hence the GS is $y^3\,(1+x)^2 = \frac{x^4}{4} + \frac{2x^5}{5} + \frac{x^6}{6} + c$

5. $Solve\ (1-x^2)\frac{dy}{dx} + xy = xy^2$

Solution: The given differential equation is $(1-x^2)\frac{dy}{dx} + xy = xy^2$

Dividing with $(1-x^2)y^2$ on both sides

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{xy}{(1-x^2)y^2} = \frac{xy^2}{(1-x^2)y^2}$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{x}{(1-x^2)y} = \frac{x}{(1-x^2)}$$

Put $\frac{1}{y} = z$ and then $-\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \frac{1}{y^2}\frac{dy}{dx} = -\frac{dz}{dx}$

Can be written as $-\frac{dz}{dx} + \frac{xz}{(1-x^2)} = \frac{x}{(1-x^2)} \Rightarrow \frac{dz}{dx} - \frac{xz}{(1-x^2)} = \frac{-x}{(1-x^2)}$

Let $P_1 = -\frac{x}{(1-x^2)}$ $\quad and \ Q_1 = \frac{-x}{(1-x^2)}$

The integrating factor (I.F) = $e^{\int P_1\, dx}$

$$= e^{\int -\frac{x}{(1-x^2)}\, dx} = e^{\frac{1}{2}\int \frac{-2x}{(1-x^2)}\, dx}$$

$$= e^{\frac{1}{2}\log(1-x^2)} = e^{\log(1-x^2)^{1/2}} = \sqrt{1-x^2} \qquad \boldsymbol{I\ F} = \sqrt{1-x^2}$$

The general solution: $z(I.F) = \int Q_1(I.F)dx + c$

$$\frac{1}{y}\sqrt{1-x^2} = \int \frac{-x}{(1-x^2)}\sqrt{1-x^2}dx + c$$

$$= \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}dx + c$$

$$= \frac{1}{2}2\sqrt{1-x^2} + c$$

The general solution $\frac{1}{y}\sqrt{1-x^2} = \sqrt{1-x^2} + c$

6. solve $(x+1)\frac{dy}{dx} + 1 = e^{x-y}$

Solution: The given differential equation is $(x+1)\frac{dy}{dx} + 1 = e^{x-y} = e^x e^{-y}$

Dividing with $(x+1)e^{-y}$ on both sides

$$e^y\frac{dy}{dx} + \frac{e^y}{x+1} = \frac{e^x}{x+1}$$

Put $e^y = z$ and then $e^y\frac{dy}{dx} = \frac{dz}{dx}$

Can be written as $\frac{dz}{dx} + \frac{z}{x+1} = \frac{e^x}{x+1}$

Let $P_1 = \frac{1}{x+1}$ $\quad and \ Q_1 = \frac{e^x}{x+1}$

The integrating factor (I.F) = $e^{\int P_1\, dx} = e^{\int \frac{1}{x+1}\, dx} = e^{\log(x+1)} = x+1$

$$\boldsymbol{I\,F = x + 1}$$

The general solution: $z(I.F) = \int Q_1(I.F)dx + c$

$$e^y(x+1) = \int \frac{e^x}{x+1}(x+1)dx + c = \int (x+1)dx$$

$$= \frac{x^2}{2} + x + c$$

∴ The general solution $e^y(x+1) = \frac{x^2}{2} + x + c$

7. *Solve* $\frac{dy}{dx} + (2x\tan^{-1}y - x^3)(1+y^2) = 0$

Solution: The given differential equation is $\frac{dy}{dx} + (2x\tan^{-1}y - x^3)(1+y^2) = 0$

Dividing with $1 + y^2$ on both sides

$$\frac{1}{1+y^2}\frac{dy}{dx} + (2x\tan^{-1}y - x^3) = 0$$

$$\frac{1}{1+y^2}\frac{dy}{dx} + 2x\tan^{-1}y = x^3$$

Put $\tan^{-1}y = z$ and then $\frac{1}{1+y^2}\frac{dy}{dx} = \frac{dz}{dx}$

Can be written as $\frac{dz}{dx} + 2xz = x^3$

Let $P_1 = 2x$ *and* $Q_1 = x^3$

The integrating factor (I.F) $= e^{\int P_1\,dx} = e^{\int 2x\,dx} = e^{x^2}$ $\boldsymbol{I\,F = e^{x^2}}$

The general solution: $z(I.F) = \int Q_1(I.F)dx + c$

$$\tan^{-1}y\,(e^{x^2}) = \int x^3(e^{x^2})dx + c$$

$$e^{x^2}\tan^{-1}y = \int x^2\,(e^{x^2})x dx + c$$

let $x^2 = t \Rightarrow 2xdx = dt \Rightarrow xdx = \frac{1}{2}dt$

$$e^{x^2}\tan^{-1}y = \int t\,(e^t)\frac{1}{2}\,dt + c = \frac{1}{2}e^t(t-1) + c$$

∴ The general solution $e^{x^2}\tan^{-1}y = \frac{1}{2}e^{x^2}(x^2 - 1) + c$

8. *Solve* $\frac{dy}{dx}(x^2 y^3 + xy) = 1$

Solution: Given that $\frac{dy}{dx}(x^2 y^3 + xy) = 1$

$\frac{dx}{dy} = x^2 y^3 + xy \Rightarrow \frac{dx}{dy} - xy = x^2 y^3$

dividing with x^2 *on both sides*

$$\frac{1}{x^2}\frac{dx}{dy} - \frac{1}{x} y = y^3 \text{ --------------- } (1)$$

Let $\frac{1}{x} = z \Rightarrow \frac{-1}{x^2}\frac{dx}{dy} = \frac{dz}{dy}$

Equation (1) can be written as $-\frac{dz}{dy} - z\,y = y^3 \Rightarrow \frac{dz}{dy} + z\,y = -y^3$

Let $P_1 = y$ *and* $Q_1 = -y^3$

The integrating factor (I.F) $= e^{\int P_1\,dx} = e^{\int y\ dy} = e^{y^2/2}$

$$\boldsymbol{I\ F = e^{y^2/2}}$$

The general solution: $z(I.F) = \int Q_1(I.F)dy + c$

$$\frac{1}{x}(e^{y^2/2}) = \int -y^3(e^{y^2/2})dy + c$$

$$\frac{1}{x}(e^{y^2/2}) = -\int y^2(e^{y^2/2})ydy + c$$

let $y^2/2 = t \Rightarrow ydy = dt$

$$\frac{1}{x}(e^{y^2/2}) = -\int 2t(e^t)dt + c = -2e^t(t-1) + c$$

$\therefore$ The general solution $\frac{1}{x}(e^{y^2/2}) = -2e^{y^2/2}\left(\frac{y^2}{2} - 1\right) + c$

9. *solve* $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$

Solution: Given that $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$

dividing with $y(\log y)^2$ *on both sides*

that $\frac{1}{y(\log y)^2}\frac{dy}{dx} + \frac{y}{xy(\log y)^2}\log y = \frac{y}{x^2 y(\log y)^2}(\log y)^2$

$$\Rightarrow \frac{1}{y(\log y)^2}\frac{dy}{dx}+\frac{1}{x\log y}=\frac{1}{x^2}$$

Let $\frac{1}{\log y}=z \Rightarrow \frac{-1}{y(\log y)^2}\frac{dy}{dx}=\frac{dz}{dx}$

Equation (1) can be written as $-\frac{dz}{dx}+\frac{z}{x}=\frac{1}{x^2} \Rightarrow \frac{dz}{dx}-\frac{z}{x}=-\frac{1}{x^2}$

Let $P_1=-\frac{1}{x}$ and $Q_1=-\frac{1}{x^2}$

The integrating factor (I.F) $= e^{\int P_1\,dx}=e^{\int -\frac{1}{x}dy}=e^{-\log x}=\frac{1}{x}$

$$\boldsymbol{I\,F=\frac{1}{x}}$$

The general solution: $z(I.F)=\int Q_1(I.F)dx+c$

$$\frac{1}{\log y}\left(\frac{1}{x}\right)=\int -\frac{1}{x^2}\left(\frac{1}{x}\right)dx+c$$

$$\frac{1}{x\log y}=-\int \frac{1}{x^3}dx+c=\frac{1}{2x^2}+c$$

Hence the general solution $\frac{1}{x\log y}=\frac{1}{2x^2}+c$

10. solve $\frac{dy}{dx}=e^{x-y}[e^x-e^y]$

Solution: The given differential equation is

$$\frac{dy}{dx}=e^{x-y}[e^x-e^y]=\frac{dy}{dx}=e^{2x-y}-e^x \Rightarrow \frac{dy}{dx}+e^x=\frac{e^{2x}}{e^y}$$

multiply with e^y on both sides

$$e^y\frac{dy}{dx}+e^ye^x=e^{2x}$$

Put $e^y=z$ and then $e^y\frac{dy}{dx}=\frac{dz}{dx}$

Can be written as $\frac{dz}{dx}+ze^x=e^{2x}$

Let $P_1=e^x$ and $Q_1=e^{2x}$

The integrating factor (I.F) $= e^{\int P_1\,dx}$

$$=e^{\int e^x\,dx}=e^{e^x} \qquad \boldsymbol{I\,F=e^{e^x}}$$

$$The\ general\ solution: z(I.F) = \int Q_1(I.F)dx + c$$

$$e^y e^{e^x} = \int e^{2x} e^{e^x} dx + c = \int e^x e^x e^{e^x} dx + c \qquad (\text{Let } e^x = t \Rightarrow e^x dx = dt)$$

$$= \int t e^t dx + c = e^t(t-1) + c$$

$\therefore$ The general solution $e^y e^{e^x} = e^{e^x}(e^x - 1) + c$

Exact and Non-Exact Differential Equations:

Definition:

A differential equation $M(x,y)dx + N(x,y)dy = 0$ is said to be exact if there exist a function $u(x,y)$ such that $M(x,y)dx + N(x,y)dy = u(x,y)$

In other words, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and the general solution can be determined by

$$\int M(x,y)dx(y\ treated\ as\ \ constent) + \int N(x,y)dy(Remove\ x - term) = c$$

Problems:

1. $Solve\ (x + 2y - 3)dy - (2x - y + 1)dx = 0$

Solution: $The given differential equation can be written as$

$$(2x - y + 1)dx - (x + 2y - 3)dy = 0$$

Let $M = 2x - y + 1$ and $N = -(x + 2y - 3)$

Now $\frac{\partial M}{\partial y} = -1$ *and* $\frac{\partial N}{\partial x} = -1$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and is Exact

The general solution is

$$\int M(x,y)dx(ytreatedasconstent) + \int N(x,y)dy(Removex - term) = c$$

$$\int (2x - y + 1)dx(ytreatedasconstent) - \int (2y - 3)dy(Removex - term) = c$$

$$\Rightarrow 2\frac{x^2}{2} - xy + x - 2\frac{y^2}{2} + 3y = c$$

$\Rightarrow$ The general solution $\boldsymbol{x^2 - y^2 - xy + x = c}$

2.$Solve (4x+3y+1)dx+(3x+2y+1)dy=0$

Solution:$The given differential equation$ $(4x+3y+1)dx+(3x+2y+1)dy=0$

Let $M=(4x+3y+1)$ and $N=(3x+2y+1)$

Now $\frac{\partial M}{\partial y}=3$ $and \frac{\partial N}{\partial x}=3$ $\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ and is Exact

The general solution is

$\int M(x,y)dx(y-treated\ as\ constent)+\int N(x,y)dy(Remove\ x\ term)=c$

$$\Rightarrow \int (4x+3y+1)dx+\int 2y+1)dy=c$$

$$\Rightarrow 4\frac{x^2}{2}+3xy+x+2\frac{y^2}{2}+y=c$$

$\Rightarrow$The general solution $\boldsymbol{2x^2+y^2+3xy+x+y=c}$

3. Solve $\frac{dy}{dx}=\frac{x^2-4xy-2y^2}{2x^2+4xy-y^2}$

Solution: $The given differential equation can be written as$

$$(2x^2+4xy-y^2)dy=(x^2-4xy-2y^2)dx$$

That is $(x^2-4xy-2y^2)dx-(2x^2+4xy-y^2)dy=0$

Let $M=x^2-4xy-2y^2$ and $N=-(2x^2+4xy-y^2)$

Now $\frac{\partial M}{\partial y}=-4x-4y$ $and \frac{\partial N}{\partial x}=-4x-4y$

$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$Exactdifferential equation.

The general solution is

$$\int M(x,y)dx(y\ treated\ as\ constent)+\int N(x,y)dy(Remove\ x\ term)=c$$

$$\Rightarrow \int (x^2-4xy-2y^2)dx+\int -(-y^2)dy=c$$

$$\Rightarrow \frac{x^3}{3}-4\frac{x^2}{2}y-2xy^2+\frac{y^3}{3}=c$$

$$\Rightarrow \frac{x^3}{3}-2x^2y-2xy^2+\frac{y^3}{3}=c$$

$\Rightarrow$ The general solution $x^3-6x^2y-6xy^2+y^3=c$

4. $Solve: (1+ e^{x/y})dx + (1-\frac{x}{y})e^{x/y}dy = 0$

Solution: $The given differential equation\ (1+ e^{x/y})dx + (1-\frac{x}{y})e^{x/y}dy = 0$

Let $M = (1+e^{x/y})$ and $N = (1-\frac{x}{y})e^{x/y}$

Now $\frac{\partial M}{\partial y} = 0 + e^{\frac{x}{y}}(-\frac{x}{y^2}) = e^{\frac{x}{y}}(-\frac{x}{y^2})$ and

$\frac{\partial N}{\partial x} = \frac{-1}{y}e^{x/y} + (1-\frac{x}{y})\ e^{\frac{x}{y}}(\frac{1}{y}) = -\frac{1}{y}e^{\frac{x}{y}}[\ 1-1+\frac{x}{y}] = e^{\frac{x}{y}}(-\frac{x}{y^2})$

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and exact

The general solution is

$$\int M(x,y)dx(y\ treated\ as\ constent) + \int N(x,y)dy(Remove x\ term) = c$$

$$\Rightarrow \int (1+ e^{\frac{x}{y}})\,dx + 0 = c$$

$\Rightarrow x + \frac{e^{\frac{x}{y}}}{\frac{1}{y}} = c \quad \Rightarrow$ The general solution $x + ye^{\frac{x}{y}} = c$

5. $solve\ (e^y + 1)\cos x\ dx + e^y \sin x\ dy = 0$

Solution: The given differential equation

$$(e^y + 1)\cos x\ dx + e^y \sin x\ dy = 0$$

Let $M = (e^y + 1)Cos\ x$ and $N = e^y sinx$

Now $\frac{\partial M}{\partial y} = e^y Cos\ x$ and $\frac{\partial N}{\partial x} = e^y\ Cos\ x$ $\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and exact

The general solution is

$$\int M(x,y)dx(y\ treated\ as\ constent) + \int N(x,y)dy(Remove x\ term) = c$$

$$\int (e^y + 1)Cos\ x\ dx \quad = c$$

The general solution $(e^y + 1)sin\ x\ = \mathrm{c}$

6. $solve\ \frac{dy}{dx} + \frac{ax+hy+g}{hx+by+f} = 0$

Solution: The given differential equation can be written as

$$(hx + by + f)\, dy + (ax + hy + g)dx = 0$$

That is $(ax + hy + g)dx + (hx + by + f)\, dy = 0$

Let $M = ax + hy + g$ and $N = hx + by + f$

Now $\frac{\partial M}{\partial y} = h$ *and* $\frac{\partial N}{\partial x} = h$ $\quad \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ *and exact*

The general solution is

$$\int M(x,y)dx(y \text{ treated as constent}) + \int N(x,y)dy(\text{Removex term}) = c$$

$$\int (ax + hy + g)\, dx + \int (by + f)\, dy = c$$

$$a\frac{x^2}{2} + hxy + gx + b\frac{y^2}{2} + fy = c$$

The general solution $ax^2 + 2hxy + by^2 + 2gx + 2fy = 2c$

$Non -\ exact\ differential\ equations$:

A differential equation $M(x,y)dx + N(x,y)dy = 0$ is said to be non-exact if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

They are four types of non-exact differential equations

Type-1 Reduce into exact using the Integrating factors.

The following formulae are use to solve such non-exact differential equations.

1. $2xdx + 2ydy = d\,[x^2 + y^2]$

2. $xdy + ydx = d(xy)$

3. $\frac{ydx - xdy}{y^2} = d[\frac{x}{y}]$

4. $\frac{xdy - ydx}{x^2} = d[\frac{y}{x}]$

5. $-\frac{xdy + ydx}{x^2y^2} = d\,[\frac{1}{xy}]$

6. $\frac{ydx - xdy}{xy} = d[\log \frac{x}{y}]$

7. $\frac{xdy - ydx}{xy} = d\,[\log \frac{y}{x}]$

8. $\frac{ydx - xdy}{x^2+y^2} = d[\tan^{-1} x/y]$

9. $\frac{xdy - ydx}{x^2+y^2} = d[\tan^{-1} y/x]$

10. $\frac{ye^x dx - e^x dy}{y^2} = d[\frac{e^x}{y}]$

11. $\frac{xe^y y - e^y dx}{x^2} = d[\frac{e^y}{x}]$

Problems:

1. Solve $xdy - ydx = xy^2dx$

Solution: The given differential equation can be written as

$$\frac{xdy-ydx}{y^2} = x\,dx \;\Rightarrow\; \frac{-(ydx-xdy)}{y^2} = x\,dx$$

$$\Rightarrow -d\left[\frac{x}{y}\right] = x\,dx$$

$$\Rightarrow \quad -\int d\left[\frac{x}{y}\right] = \int x\,dx + c$$

$$\Rightarrow \frac{-x}{y} = \frac{x^2}{2} + c \quad \Rightarrow \text{The general solution } \frac{x}{y} + \frac{x^2}{2} + c = 0$$

2. *Solve* $(1+xy)xdy + (1-xy)ydx = 0$

Solution: The given differential equation can be written as

$$xdy + x^2y\,dy + ydx - xy^2dx = 0$$

$$\Rightarrow x\,dy + y\,dx + xy(xdy - ydx) = 0$$

Dividing with x^2y^2 *on both sides*

$$\Rightarrow \frac{xdy+ydx}{x^2y^2} + \frac{xy(xdy-ydx)}{x^2y^2} = 0$$

$$\Rightarrow \frac{xdy+ydx}{x^2y^2} + \frac{(xdy-ydx)}{xy} = 0$$

$$\Rightarrow d\left[\frac{1}{xy}\right] + d\left[\log\frac{y}{x}\right] = 0$$ Integrating on both sides

$$\Rightarrow \int d\left[\frac{1}{xy}\right] + \int d\left[\log\frac{y}{x}\right] = c$$

$$\Rightarrow \text{The general solution } \frac{1}{xy} + \log\frac{y}{x} = c$$

3. Solve $xdx + ydy + \dfrac{xdy-ydx}{x^2+y^2} = 0$

Solution: Given that $xdx + ydy + \dfrac{xdy-ydx}{x^2+y^2} = 0$

$$\Rightarrow \int xdx + \int ydy + \int \frac{xdy-ydx}{x^2+y^2} = c$$

$$\Rightarrow \int xdx + \int ydy + \int d[\tan^{-1} y/x] = c$$

The general solution $\dfrac{x^2}{2} + \dfrac{y^2}{2} + \tan^{-1} y/x = c$

4. $Solve\ \ ydx - xdy + \log x\ dx = 0.$

Solution: Given that $\quad ydx - xdy + \log x\ dx = 0$

Dividing with x^2 on both sides

$$\frac{ydx-xdy}{x^2} + \frac{\log x}{x^2}\ dx = c$$

$$\Rightarrow \int \frac{-(xdy-ydx)}{x^2} + \int \frac{\log x}{x^2}\ dx = c$$

$$\Rightarrow -\int d\left(\frac{y}{x}\right) + \int \log x\ \frac{1}{x^2}\ dx = c$$

$$\Rightarrow -\frac{y}{x} + \left[\log x\ \left(-\frac{1}{x}\right) - \int \frac{1}{x}\left(-\frac{1}{x}\right) dx\right] = c$$

$$\Rightarrow -\frac{y}{x} + \left[-\frac{1}{x}\log x\ + \int \frac{1}{x^2}\ dx\right] = c$$

$$\Rightarrow -\frac{y}{x} - \frac{1}{x}\log x\ - \frac{1}{x} = c$$ $\Rightarrow$The general solution $\frac{y}{x} + \frac{1}{x}\log x\ + \frac{1}{x} = c$

5. Solve $\quad xdy = [y + xcos^2\ (\frac{y}{x})]$ d x

Solution: The given differential equation can be written as

$$xdy - ydx = xcos^2\ \left(\frac{y}{x}\right)\ dx$$

$$\Rightarrow \frac{xdy-ydx}{cos^2\left(\frac{y}{x}\right)} = x\ dx$$

$$\Rightarrow sec^2\frac{y}{x}\ \left[\frac{xdy-ydx}{x^2}\right] = \frac{x\ dx}{x^2}$$

$$\Rightarrow \int sec^2\frac{y}{x}\ d\left(\frac{y}{x}\right) \quad = \int \frac{1}{x}\ dx$$

$\Rightarrow$ The general solution $\tan \frac{y}{x} = \log x + c$

6. $Solve\ (x^2 + y^2 + x)dx - (2x^2 + 2y^2 - y)dy = 0$

Solution: The given differential equation can be written as

$$(x^2 + y^2)dx + xdx - 2(x^2 + y^2) + ydy = 0$$

$$\Rightarrow [dx - 2dy] + xdx + ydy = 0$$

$$\Rightarrow\ dx - 2dy + \frac{2(xdx+ydy)}{2(x^2+y^2)} = 0$$

$$\Rightarrow\ \int dx\ - 2\int \mathrm{dy}\ + \frac{1}{2}\int \frac{2xdx+2ydy}{(x^2+y^2)}\ = \mathrm{c}$$

$\Rightarrow$ The general solution $x - 2y + \frac{1}{2}\log(x^2 + y^2) = \mathrm{c}$

Type-2 Homogeneous non-exact Differential equation

A non-exact differential equation is in the form $M(x,y)dx + N(x,y)dy = 0$ where $M(x,y), N(x,y)$ are homogeneous equations the to solve this equation when we reduce into exact using an Integrating Factor $\frac{1}{Mx+Ny}$ $where\ Mx + Ny \neq 0$.

Problems:

1. Solve $x^2y\ dx - (\ x^3 +\ y^3\ )dy = 0$

Solution: In the given differential equation $x^2y\ dx - (\ x^3 +\ y^3\ )dy = 0$

Let M = x^2y N = $-(\ x^3 +\ y^3\ )$

$$\frac{\partial M}{\partial y} =\ x^2 \qquad \frac{\partial N}{\partial x} = -3x^2 \qquad \therefore\ \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\ \text{ not exact}$$

And is homogeneous the integrating factor = $\frac{1}{Mx+Ny} = \frac{-1}{y^4}$

(As $Mx + Ny = (x^2y)x - (\ x^3 +\ y^3\ )y =\ x^3y -\ x^3y -\ y^4 = -\ y^4$)

Multiply the integrating factor with the given DE

$$-\frac{x^2y}{y^4}\ dx + \frac{x^3+y^3}{y^4}dy = 0\ \Rightarrow -\ \frac{x^2}{y^3}\ dx + [\ \frac{x^3}{y^4} + \frac{1}{y}]dy = 0$$

Let M = $-\ \frac{x^2}{y^3}$ $N =\ \frac{x^3}{y^4} + \frac{1}{y}$

$$\frac{\partial M}{\partial y} = \ -x^2\ [-\frac{3}{y^4}] = \frac{3x^2}{y^4} \qquad \frac{\partial N}{\partial x}\ = \frac{3x^2}{y^4} + 0 = \frac{3x^2}{y^4}$$

$$\therefore\ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\quad \text{exact}$$

The general solution $\int -\ \frac{x^2}{y^3}\ dx + \int \frac{1}{y}\ dy = c$

$$\Rightarrow \frac{-1}{y^3}\frac{x^3}{3} + \log y = c$$

$\Rightarrow$ The general solution $\frac{-x^3}{3y^3} + \log y = c$

2. Solve $y^2\,dx - (\,x^2 - \,y^2 - xy)dy = 0$

Solution: In the given differential equation $y^2\,dx + (\,x^2 - \,y^2 - xy)dy = 0$

Let M = y^2 N = $(\,x^2 - \,y^2 - xy)$

$\frac{\partial M}{\partial y} = \,2y$ $\frac{\partial N}{\partial x} = 2x - y$ $\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ not exact

And is homogeneous the integrating factor = $\frac{1}{Mx+Ny} = \frac{1}{y(x^2 - \,y^2)}$

(As $Mx + Ny = (y^2)x + (\,x^2 - \,y^2 - xy\,)y$

$= xy^2 + \,x^2y - \,y^3 - xy^2 = \,x^2y - \,y^3$)

$= \,y(x^2 - \,y^2)$

Multiply the integrating factor with the given DE

$$\frac{y^2}{y(x^2 - \,y^2)}dx + \frac{x^2 - \,y^2 - xy}{y(x^2 - \,y^2)}\;dy = 0$$

Let M = $\frac{y}{(x^2 - \,y^2)}$ $N = \frac{x^2 - \,y^2 - xy}{y(x^2 - \,y^2)} = \frac{x^2 - \,y^2}{y(x^2 - \,y^2)} - \frac{xy}{y(x^2 - \,y^2)} = \frac{1}{y} - \frac{x}{x^2 - \,y^2}$

$\frac{\partial M}{\partial y} = \frac{(x^2 - \,y^2) - y(0 - 2y)}{(x^2 - \,y^2)^2} = \frac{(x^2 + \,y^2)}{(x^2 - \,y^2)^2}$ $\frac{\partial N}{\partial x} = \;0 - \frac{(\,x^2 - \,y^2) - x(2x - 0)}{(x^2 - \,y^2)^2} = \frac{(x^2 + \,y^2)}{(x^2 - \,y^2)^2}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ exact

The general solution $\int \frac{y}{(x^2 - \,y^2)}\;dx + \int \frac{1}{y}\;dy = c$

$y[\,\frac{1}{2y}\,\log\frac{x-y}{x+y}] + \log y = c \Rightarrow$ The general solution $\frac{1}{2}\,\log\frac{x-y}{x+y} + \log y = c$

3. Solve $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy \,= 0$

Solution: In the given differential equation

Let M = $x^2y - 2xy^2$ N = $-(x^3 - 3x^2y)$

$$\frac{\partial M}{\partial y} = x^2 - 4xy \qquad \frac{\partial N}{\partial x} = -(3x^2 - 6xy) \qquad \therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ not exact}$$

And is homogeneous the integrating factor = $\frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$

(As $Mx + Ny = (x^2y - 2xy^2)x - (x^3 - 3x^2y)y = x^3y - 2x^2y^2 - x^3y + 3x^2y^2 = x^2y^2$)

Multiply the integrating factor with the given DE

$$\frac{x^2y-2xy^2}{x^2y^2}dx - \frac{(x^3-3x^2y}{x^2y^2}dy = 0$$

$$[\frac{1}{y} - \frac{2}{x}]\,dx - [\frac{x}{y^2} - \frac{3}{y}]dy = 0$$

Let M = $\frac{1}{y} - \frac{2}{x}$ $N = -\frac{x}{y^2} + \frac{3}{y}$

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2} - 0 = -\frac{1}{y^2} \qquad \frac{\partial N}{\partial x} = -\frac{1}{y^2} - 0 = -\frac{1}{y^2}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ exact}$$

The general solution $\int[\frac{1}{y} - \frac{2}{x}]$ dx $-\int \frac{-3}{y}dy = c$

$\Rightarrow$ The general solution $\frac{x}{y} - 2\log x + 3\log y = c$

Type-3 *Non* $-$ *exact* $yf(x,y)dx + xf(x,y)dy = 0$ *form*

A non-exact differential equation is in the form $M(x,y)dx + N(x,y)dy = 0$ or

$yf(x,y)dx + xf(x,y)dy = 0$ the to solve this equation when we reduce into exact using an Integrating Factor $\frac{1}{Mx-Ny}$ $where\ Mx - Ny \neq 0$.

Problems: 1.Solve $y(1 + xy)dx + x(1 - xy)dy = 0$

Solution: *In the given differential equation*

$M = y(1+xy) = y + xy^2$ $\quad N = x(1-xy) = x - x^2y$

$\frac{\partial M}{\partial y} = 1 + 2xy$ $\quad \frac{\partial N}{\partial x} = 1 - 2xy$ $\quad \therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ not exact.

$$The\ integrating\ factor = \frac{1}{Mx - Ny} = \frac{1}{2\,x^2y^2}$$

$(\ since\ \ Mx - Ny = (y + xy^2)x - (x - x^2y)y = xy + x^2y^2 - xy + x^2y^2 = 2\,x^2y^2 \neq 0)$

Multiply the integrating factor with the given DE

$$\frac{y + xy^2}{2\,x^2y^2}dx + \frac{x - x^2y}{2\,x^2y^2}dy = 0$$

$$[\frac{1}{2yx^2} + \frac{1}{2x}]\,dx + [\frac{1}{2xy^2} - \frac{1}{2y}]dy = 0$$

Let $M = \frac{1}{2yx^2} + \frac{1}{2x}$ $\quad N = \frac{1}{2xy^2} - \frac{1}{2y}$

$\frac{\partial M}{\partial y} = -\frac{1}{2y^2x^2}$ $\quad \frac{\partial N}{\partial x} = -\frac{1}{2y^2x^2}$ $\quad \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ exact

The general solution $\int \frac{1}{2yx^2} + \frac{1}{2x}\ dx + \int \frac{1}{2xy^2} - \frac{1}{2y}\ dy = c$

$\Rightarrow$ The general solution $\frac{-1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = c$

2. Solve $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$

Solution: $In\ the\ given\ differential\ equation$

$M = y(xy + 2x^2y^2) = xy^2 + 2x^2y^3$ $\quad N = x(xy - x^2y^2) = x^2y - x^3y^2$

$\frac{\partial M}{\partial y} = 2xy + 6\,x^2y^2$ $\quad \frac{\partial N}{\partial x} = 2xy - 3x^2y^2$ $\quad \therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ not exact.

$$The\ integrating\ factor = \frac{1}{Mx - Ny} = \frac{1}{3x^3y^3}$$

$(\ since\ Mx - Ny = (xy^2 + 2x^2y^3)x - (x^2y - x^3y^2)y$

$$= x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3 = 3x^3y^3 \neq 0)$$

Multiply the integrating factor with the given DE

$$\frac{xy^2 + 2x^2y^3}{3x^3y^3}dx + \frac{x^2y - x^3y^2}{3x^3y^3}dy = 0$$

$$[\frac{1}{3x^2y} + \frac{2}{3x}]\,dx + [\frac{1}{3xy^2} - \frac{1}{3y}]dy = 0$$

Let $M = \frac{1}{3x^2y} + \frac{2}{3x}$ $\qquad$ $N = \frac{1}{3xy^2} - \frac{1}{3y}$

$\frac{\partial M}{\partial y} = -\frac{1}{3y^2x^2}$ $\qquad$ $\frac{\partial N}{\partial x} = -\frac{1}{3y^2x^2}$ $\quad \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ exact

The general solution $\int \frac{1}{3x^2y} + \frac{2}{3x}\ dx + \int \frac{1}{3xy^2} - \frac{1}{3y}\ dy = c$

$\Rightarrow$ The general solution $\frac{-1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c$

3.. *Solve* $x(1+xy)dy + y(1-xy)dx = 0$

Solution: *The given DE can be written as* $y(1-xy)dx + x(1+xy)dy = 0$

In the given differential equation

$M = y(1-xy) = y - xy^2$ $\qquad$ $N == x(1+xy) = x + yx^2$

$\frac{\partial M}{\partial y} = 1 - 2xy$ $\qquad$ $\frac{\partial N}{\partial x} = 1 + 2xy$ $\quad \therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ not exact.

The integrating factor $= \frac{1}{Mx - Ny} = \frac{-1}{2\,x^2y^2}$

(*since* $Mx - Ny = (y - xy^2)x - (x + x^2y)y = xy - x^2y^2 - xy - x^2y^2 = -2\,x^2y^2 \neq 0$)

Multiply the integrating factor with the given DE

$$\frac{y - xy^2}{-2\,x^2y^2}dx + \frac{x + x^2y}{-2\,x^2y^2}dy = 0$$

$$\left[\frac{1}{2yx^2} - \frac{1}{2x}\right]dx + \left[\frac{1}{2xy^2} + \frac{1}{2y}\right]dy = 0$$

Let $M = \frac{1}{2yx^2} - \frac{1}{2x}$ $\qquad$ $N = \frac{1}{2xy^2} + \frac{1}{2y}$

$\frac{\partial M}{\partial y} = -\frac{1}{2y^2x^2}$ $\qquad$ $\frac{\partial N}{\partial x} = -\frac{1}{2y^2x^2}$ $\quad \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ exact

The general solution $\int \frac{1}{2yx^2} - \frac{1}{2x}\ dx + \int \frac{1}{2xy^2} + \frac{1}{2y}\ dy = c$

$\Rightarrow$ The general solution $\frac{-1}{2xy} - \frac{1}{2}\log x + \frac{1}{2}\log y = c$

Type-4 $\quad$ *Non − exact* $Mdx + Ndy = 0$ *form*

A non-exact differential equation is in the form $M(x,y)dx + N(x,y)dy = 0$ or

$M(x,y)dx + N(x,y)(\text{least function copare to } M)dy = 0$ the to solve this equation when we reduce into exact using an Integrating Factor $e^{\int f(x)dx}$ where

$$f(x) = \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]$$

Problems: 1. Solve $2xydy - (x^2 + y^2 + 1)dx = 0$

Solution: The given differential equation an be written as

$$(x^2 + y^2 + 1)dx - 2xydy = 0$$

Let M = $x^2 + y^2 + 1$ $\qquad N = -2xy$

$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = -2y \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ *non exact DE*

Clearly N is a least function $\quad \therefore\ f(x) = \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{-2xy}[2y + 2y] = \frac{1}{-2xy}[4y] = \frac{-2}{x}$

Integrating factor, I F = $e^{\int f(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = \frac{1}{x^2}$

$$\text{I F} = \frac{1}{x^2}$$

Multiply the IF with the given DE we get

$$\left[\frac{x^2+y^2+1}{x^2}\right]dx - \frac{2xy}{x^2}dy = 0$$

$$\left[1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right]dx - \frac{2y}{x}dy = 0$$

Let $M_1 = 1 + \frac{y^2}{x^2} + \frac{1}{x^2} \qquad N_1 = -\frac{2y}{x}$

$\frac{\partial M_1}{\partial y} = 0 + \frac{2y}{x^2} + 0 = \frac{2y}{x^2} \qquad \frac{\partial N_1}{\partial x} = -2y\left(-\frac{1}{x^2}\right) = \frac{2y}{x^2} \qquad \therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ *exact DE*

The general solution is $\quad \int\left[1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right]dx + 0 = c$

$\Rightarrow x - \frac{y^2}{x} - \frac{1}{x} = c \quad \Rightarrow$ The general solution $x^2 - y^2 - 1 = cx$

Problems: 2. Solve $(x^2 + y^2 + 2x)dx + 2ydy = 0$

Solution: The given differential equation is $\quad (x^2 + y^2 + 2x)dx + 2ydy = 0$

Let M = $x^2 + y^2 + 2x$ $\qquad N = 2y$

$$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 0 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \; non\ exact\ DE$$

Clearly N is a least function $\therefore \; f(x) = \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{2y}\left[2y - 0\right] = 1$

Integrating factor, I F = $e^{\int f(x)dx} = e^{\int 1\,dx} = e^{x}$

$$\text{I F} = e^{x}$$

Multiply the IF with the given DE we get

$$e^{x}(x^2 + y^2 + 2x)dx + 2ye^{x}dy = 0$$

Let $M_1 = e^{x}(x^2 + y^2 + 2x)$ $\qquad N_1 = 2ye^{x}$

$$\frac{\partial M_1}{\partial y} = e^{x}(2y) = 2ye^{x} \qquad \frac{\partial N_1}{\partial x} = 2ye^{x} \qquad \therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \quad exact\ DE$$

The general solution is $\int e^{x}(x^2 + y^2 + 2x)dx + 0 = c$ using Integration by parts

$$e^{x}(x^2 + y^2 + 2x) - e^{x}(2x + 2) + e^{x}(2) = c$$

$$e^{x}(x^2 + y^2 + 2x - 2x - 2 + 2) = c$$

$\Rightarrow$ The general solution $e^{x}(x^2 + y^2) = c$

Problems: 3. Solve $(1 + y + x^2y)dx + (x + x^3)dy = 0$

Solution: The given differential equation is $(1 + y + x^2y)dx + (x + x^3)dy = 0$

Let M = $1 + y + x^2y$ $\qquad N = x + x^3$

$$\frac{\partial M}{\partial y} = 1 + x^2 \qquad \frac{\partial N}{\partial x} = 1 + 3x^2 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \; non\ exact\ DE$$

Clearly N is a least function $\therefore \; f(x) = \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{x+x^3}\left[1 + x^2 - 1 - 3x^2\right]$

$$= \frac{-2x^2}{x(1 + x^2)} = \frac{-2x}{(1 + x^2)}$$

Integrating factor, I F = $e^{\int f(x)dx} = e^{\int \frac{-2x}{(1+x^2)}dx} = e^{-\log(1+x^2)}$

$$\text{I F} = \frac{1}{1+x^2}$$

Multiply the IF with the given DE we get

$$\frac{1+y+x^2y}{1+x^2}\,dx + \frac{x+x^3}{1+x^2}\,dy = 0 \;\Rightarrow\; \left(\frac{1}{1+x^2}+y\right)dx + x\,dy = 0$$

Let $M_1 = \frac{1}{1+x^2} + y$ $\qquad N_1 = x$

$\frac{\partial M_1}{\partial y} = 1 \qquad \frac{\partial N_1}{\partial x} = 1 \qquad \therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ $\quad$ *exact DE*

The general solution is $\int \frac{1}{1+x^2} + y \;\; dx + 0 = c$ using Integration by parts

$\Rightarrow$ The general solution $\tan^{-1} x + xy = c$

Type-5 $\quad$ *Non − exact $Mdx + Ndy = 0$ form*

A non-exact differential equation is in the form $M(x, y)dx + N(x, y)dy = 0$ or

$M(x, y)$(*least function copare to N*)$dx + N(x, y)dy = 0$ the to solve this equation when we reduce into exact using an Integrating Factor $e^{\int g(y)dy}$ where

$$g(y) = \frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]$$

Problems:

1.Solve $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$

Solution: The given differential equation is $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$

Let M = $xy^3 + y$ $\qquad N = 2(x^2y^2 + x + y^4)$

$\frac{\partial M}{\partial y} = 1 + 3xy^2 \qquad \frac{\partial N}{\partial x} = 2(2xy^2 + 1) = 4xy^2 + 2 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ *non exact DE*

Clearly M is a least function $\quad \therefore g(y) = \frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{1}{xy^3+y}\left[4xy^2 + 2 - 1 - 3xy^2\right]$

$$= \frac{1}{y(xy^2 + 1)}\{xy^2 + 1\} = \frac{1}{y}$$

$I\,F = e^{\int g(y)dy} = e^{\int \frac{1}{y}dy} = e^{\log y} = y$ $\qquad$ I F = y

Multiply the IF with the given DE we get

$$(xy^4 + y^2)dx + 2(x^2y^3 + xy + y^5)dy = 0$$

Let $M_1 = xy^4 + y^2$ $\qquad N_1 = 2(x^2y^3 + xy + y^5)$

$$\frac{\partial M_1}{\partial y} = 4xy^3 + 2y \qquad \frac{\partial N_1}{\partial x} = 4xy^3 + 2y \qquad \therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \quad exact\ DE$$

The general solution is

$$\int xy^4 + y^2\, dx + \int 2\, y^5\ dy + 0 = c$$

$$\frac{x^2y^4}{2} + xy^2 + 2\ \frac{y^6}{6} = c$$

$\Rightarrow$ The general solution $3x^2y^4 + 6xy^2 + 2y^6 = c$

2. Solve $(y^4 + 2y)dx + (xy^3 - 4x + 2y^4)dy = 0$

Solution: The given differential equation is $(y^4 + 2y)dx + (xy^3 - 4x + 2y^4)dy = 0$

Let M = $y^4 + 2y$ $\qquad N = xy^3 - 4x + 2y^4$

$$\frac{\partial M}{\partial y} = 4y^3 + 2 \qquad \frac{\partial N}{\partial x} = (y^3 - 4 + 0) = y^3 - 4 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\ non\ exact\ DE$$

Clearly M is a least function $\quad \therefore g(y) = \frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]$

$$= \frac{1}{y^4+2y}\left[y^3 - 4 - 4y^3 - 2\right]$$

$$= \frac{1}{y(y^3+2)}\{-3y^3 - 6\} = \frac{-3(y^3+2)}{y(y^3+2)} = \frac{-3}{y}$$

$$I\,F = e^{\int g(y)dy} = e^{\int \frac{-3}{y}dy} = e^{-3\log y} = \frac{1}{y^3} \qquad \text{I F} = \frac{1}{y^3}$$

Multiply the IF with the given DE we get

$$\frac{y^4+2y}{y^3}\, dx + \frac{xy^3-4x+2y^4}{y^3}\, dy = 0 \Rightarrow \left[y + \frac{2}{y^2}\right]dx + \left[x - \frac{4x}{y^3} + 2y\right]dy = 0$$

Let $M_1 = y + \frac{2}{y^2} \qquad N_1 = x - \frac{4x}{y^3} + 2y$

$$\frac{\partial M_1}{\partial y} = 1 - \frac{4}{y^3} \qquad \frac{\partial N_1}{\partial x} = 1 - \frac{4}{y^3} \qquad \therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \quad exact\ DE$$

The general solution is $\quad \int\left[y + \frac{2}{y^2}\right]dx + \int 2y\ \ dy = c \Rightarrow xy + \frac{2x}{y^2} + y^2 = c$

The general solution $xy + \frac{2x}{y^2} + y^2 = c$

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